ECE 307 – Techniques for Engineering Decisions

12. Probability Distributions

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SCOPE OF LECTURE

- We review basic probability distributions
- The entire lecture is simply a review of known probability material given the prerequisites
- We extensively rely on examples to drive home the usefulness of the material
- We rely on the use of tabulated data of probability distributions

OUTLINE OF DISTRIBUTION REVIEWED

- ❑ Discrete distributions
    - ❍ binomial
    - ❍ Poisson
- ❑ Continuous distributions
    - ❍ exponential
    - ❍ normal

THE BINOMIAL DISTRIBUTION

- **Binomial distributions are used to describe events with only two possible outcomes**
- **Basic requirements are**
  - *dichotomous outcomes*: **uncertain events occur in a sequence with each event having one of two possible outcomes such as:**

THE BINOMIAL DISTRIBUTION

- success/failure
- on/off
- correct/incorrect
- true/false

- *constant probability*: each event has the same probability of success
- *independence*: the outcome of each event is independent of the outcomes of any other event

BINOMIAL DISTRIBUTION EXAMPLE

- ☐ We consider a group of n identical machines with each machine having one of two states:

$$P\left\{machine\ \ is\ \ on\right\} = p$$

$$P\left\{machine\ is\ off\right\} = q = 1 - p$$

- ☐ For concreteness, let us set $n = 8$ and define for $i = 1, 2, \ldots, 8$, the $r.v.$ s :

BINOMIAL DISTRIBUTION EXAMPLE

$$\underset{\sim}{X}_i = \begin{cases} 1 & \textit{machine i is on with prob.} \quad p \\ 0 & \textit{machine i is off with prob.} \quad q = 1 - p \end{cases}$$

- ☐ The probability that 3 or more machines are on is determined by the evaluation of the probability

$$P\left\{ \sum_{i=1}^{8} \underset{\sim}{X}_i \geq 3 \right\} = P\left\{ 3 \textit{ or more machines are on} \right\}$$

BINOMIAL DISTRIBUTION EXAMPLE

$$= \; P\left\{ 3 \;\; machines \; are \; on \right\} \; +$$

$$P\left\{ 4 \;\; machines \; are \; on \right\} \; +$$

$$\ldots \qquad +$$

$$P\left\{ 8 \; machines \; are \; on \right\}$$

$$P\left\{ \sum_{i=1}^{8} \underset{\sim}{X}_i \; \geq \; 3 \right\} \; = \; \sum_{r=3}^{8} \frac{8!}{\left(8-r\right)!r!} \, p^{r} \left(1-p\right)^{8-r}$$

THE BINOMIAL DISTRIBUTION

- ☐ In general, for a *r.v.* $\underset{\sim}{R}$ with dichotomous outcomes of success and failure, the probability of *r* successes in *n* trials is

$$P\left\{\underset{\sim}{R} = r \ \textit{in} \ n \ \textit{trials with probability of success} \ p\right\}$$

$$= \frac{n!}{\left(n-r\right)!\,r!}\, p^{r}\left(1-p\right)^{n-r}$$

the binomial distribution

THE BINOMIAL DISTRIBUTION

- We can show that:

$$E\left\{\underset{\sim}{R}\right\} = n\,p$$

$$var\left\{\underset{\sim}{R}\right\} = n\,p\left(1-p\right)$$

$$P\left\{\sum_{i=1}^{n}\underset{\sim}{X}_{i} \geq k\right\} = \sum_{r=k}^{n}\frac{n!}{\left(n-r\right)!\;r!}\,p^{r}\left(1-\mathrm{p}\right)^{n-r}$$

EXAMPLE: SOFT PRETZELS

- A pretzel entrepreneur can sell each pretzel at $ 0.50 with a market potential of 100,000 pretzels within a year; as there exists a competing product, he cannot be the only seller
- Basic model is binomial:

new pretzel is a hit (*success*) ⇔ captures 30 % of market in one year

new pretzel is a flop (*failure*) ⇔ captures 10 % of market in one year

EXAMPLE: SOFT PRETZELS

- The probability of these two outcomes is equal
- Market tests are conducted with 20 pretzels taste tested against the competition; the result indicates that 5 out of 20 testers prefer the new pretzel
- We evaluate the conditional probability

$$P\left\{ new\ pretzel\ is\ a\ hit \middle| 5\ out\ of\ 20\ people\ prefer\ new\ pretzel \right\}$$

EXAMPLE: SOFT PRETZELS

- **We define the success *r.v.***

$$\underset{\sim}{S} = \begin{cases} 1 & \text{new pretzel is a hit} \quad (\text{success}) \\ 0 & \text{otherwise} \quad (\text{failure}) \end{cases}$$

with

$$P\{\underset{\sim}{S} = 1\} = P\{\underset{\sim}{S} = 0\} = 0.5$$

and

$$\underset{\sim}{X}_i = \begin{cases} 1 & \text{person } i \text{ prefers new pretzel} \\ 0 & \text{otherwise} \end{cases}$$

- **We evaluate**

$$P\left\{\text{new pretzel is a hit} \,\middle|\, 5 \text{ out of } 20 \text{ people prefer new pretzel}\right\}$$

EXAMPLE: SOFT PRETZELS

$$P\left\{ \underset{\sim}{S} = 1 \;\middle|\; \sum_{i=1}^{20} \underset{\sim}{X}_{i} = 5 \right\} = \frac{P\left\{ \underset{\sim}{S} = 1 , \sum_{i=1}^{20} \underset{\sim}{X}_{i} = 5 \right\}}{P\left\{ \sum_{i=1}^{20} \underset{\sim}{X}_{i} = 5 \right\}} =$$

$$\frac{P\left\{ \sum_{i=1}^{20} \underset{\sim}{X}_{i} = 5 \;\middle|\; \underset{\sim}{S} = 1 \right\} P\left\{ \underset{\sim}{S} = 1 \right\}}{P\left\{ \sum_{i=1}^{20} \underset{\sim}{X}_{i} = 5 \;\middle|\; \underset{\sim}{S} = 1 \right\} P\left\{ \underset{\sim}{S} = 1 \right\} + P\left\{ \sum_{i=1}^{20} \underset{\sim}{X}_{i} = 5 \;\middle|\; \underset{\sim}{S} = 0 \right\} P\left\{ \underset{\sim}{S} = 0 \right\}}$$

EXAMPLE: SOFT PRETZELS

$$\underbrace{P\left\{\sum_{i=1}^{20} \underset{\sim}{X}_i = 5 \,\middle|\, \underset{\sim}{S} = 1\right\}}_{\text{0.178 from the binomial table}}$$

is the binomial probability that 5 out of 20 people prefer the new pretzel with $p = 0.3$

$$\underbrace{P\left\{\sum_{i=1}^{20} \underset{\sim}{X}_i = 5 \,\middle|\, \underset{\sim}{S} = 0\right\}}_{\text{0.032 from the binomial table}}$$

is the binomial probability that 5 out of 20 people prefer the new pretzel with $p = 0.1$

EXAMPLE: SOFT PRETZELS

- **Therefore,**

$$\frac{P\left\{\sum_{i=1}^{20} \underset{\sim}{X}_i = 5 \middle| \underset{\sim}{S}=1\right\} P\left\{\underset{\sim}{S}=1\right\}}{P\left\{\sum_{i=1}^{20} \underset{\sim}{X}_i = 5 \middle| \underset{\sim}{S}=1\right\} P\left\{\underset{\sim}{S}=1\right\} + P\left\{\sum_{i=1}^{20} \underset{\sim}{X}_i = 5 \middle| \underset{\sim}{S}=0\right\} P\left\{\underset{\sim}{S}=0\right\}}$$

$$= \frac{(0.178)(0.5)}{(0.178)(0.5) + (0.032)(0.5)}$$

$$= 0.848$$

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THE POISSON DISTRIBUTION

- The **binomial distribution** is appropriate for the representation of successes in repeated trials
- The **Poisson distribution** is appropriate for the representation of specific events over time, space, or some other problem–specific dimension, e.g., the number of customers who are served by a butcher in a meat market, or the number of chips judged unacceptable in a production run

REQUIREMENTS FOR A POISSON DISTRIBUTION

- **Events can happen at any of a large number of values within the range of measurement (time, space, etc.) and possibly along a continuum**

- **At a specific point *z*, *P {an event at z}* is very small and therefore events do not happen *too frequently***

REQUIREMENTS FOR A POISSON DISTRIBUTION

- ❑ Each event is independent of any other event and

$$P\left\{event\ at\ any\ point\right\}$$

is constant and *independent* of all other events

- ❑ In fact, the average number of events over a unit of measure is constant

THE POISSON DISTRIBUTED *r.v.*

- ☐ $\underset{\sim}{X}$ is the *r.v.* representing the number of events in a unit of measure

$$P\left\{\underset{\sim}{X} = k\right\} = \frac{e^{-m} m^{k}}{k!}$$

$$E\left\{\underset{\sim}{X}\right\} = m \qquad var\left\{\underset{\sim}{X}\right\} = m$$

m is the Poisson distribution parameter

- ☐ Interpretation: the Poisson distribution parameter is the mean or the variance of the distribution

EXAMPLE: POISSON DISTRIBUTION

- ☐ **Consider an assembly line for manufacturing a particular product**
  - ○ **1,024 units are produced**
  - ○ **from past experience, a flawed unit is manufactured every 197 units and so, *on average*, there are $\frac{1,024}{197} \approx 5.2$ flawed units in the 1,024 products that are are produced**

EXAMPLE: POISSON DISTRIBUTION

- Note that the Poisson conditions are satisfied
  - the sample has 1,024 units
  - there are only a few flawed units in the 1,024 sample, i.e., the event of the occurrence of a flawed unit is infrequent
  - the probability of a flawed unit is rather small
  - each flawed unit is *independent* of every other flawed unit

EXAMPLE: POISSON DISTRIBUTION

- ☐ Poisson distribution is appropriate represen– tation with $m = 5.2$ and so,

$$P\{\underset{\sim}{X} = k\} = \frac{e^{-5.2}(5.2)^{k}}{k\,!}$$

- ☐ If we want to determine the probability of 4 or more flawed units, we compute

EXAMPLE: POISSON DISTRIBUTION

$$P\left\{\underset{\sim}{X} > 4\right\} = 1 - \boxed{P\left\{\underset{\sim}{X} \leq 4\right\}} = 1 - 0.406 = 0.594$$

lookup Poisson table for $k = 4$, $m = 5.2$

- ☐ **The Poisson table states that for $k = 12$, $m = 5.2$**

$$P\left\{\underset{\sim}{X} \leq 12\right\} = 0.997$$

and therefore

$$P\left\{\underset{\sim}{X} > 12\right\} = 1 - P\left\{\underset{\sim}{X} \leq 12\right\} = 0.003$$

EXAMPLE: SOFT PRETZELS

- ❑ The pretzel enterprise is going well: several retail outlets and a street vendor are selling the pretzels
- ❑ A vendor in a new location can sell, on average, 20 pretzels per hour; the vendor in an existing location sells 8 pretzels per hour

EXAMPLE: SOFT PRETZELS

- A decision is made to try to set up a second street vendor at a different, new location
- New location is classified along three distinct categories with the given probabilities

category	*characterization*	*probability*
"good"	20 *p/h are sold*	0.7
"bad"	10 *p/h are sold*	0.2
"dismal"	6 *p/h are sold*	0.1

EXAMPLE: SOFT PRETZELS

- ☐ After the first week, a long enough period to make a mark, a 30 – minute test is run and 7 pretzels are sold during the 30 – minute test period
- ☐ We analyze the situation by defining the *r.v.*

$$\underset{\sim}{L} = \begin{cases} \text{"}good\text{"} & 10 & p.\ sold\ during\ test\ \ period \\ \text{"}bad\text{"} & 5 & p.\ sold\ during\ test\ \ period \\ \text{"}dismal\text{"} & 3 & p.\ sold\ during\ test\ \ period \end{cases}$$

and assume Poisson distribution applies

EXAMPLE: SOFT PRETZELS

- ☐ **We determine the conditional probabilities of the new location conditioned on the 30-*minute* test outcomes and evaluate**

  $P\left\{ \underset{\sim}{L} = \text{"}good\text{"} \middle| \underset{\sim}{X} = 7 \right\}, \; P\left\{ \underset{\sim}{L} = \text{"}bad\text{"} \middle| \underset{\sim}{X} = 7 \right\}$ **and**

  $P\left\{ \underset{\sim}{L} = \text{"}dismal\text{"} \middle| \underset{\sim}{X} = 7 \right\}$

- ☐ **We compute the values of the Poisson distributed**

$$P\left\{ \underset{\sim}{X} = 7 \middle| \underset{\sim}{L} = \text{"}good\text{"} \right\} = \frac{e^{-10}\left(10\right)^{7}}{7!} = 0.09$$

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EXAMPLE: SOFT PRETZELS

$$P\left\{\underset{\sim}{X}=7 \,\middle|\, \underset{\sim}{L}=\text{"}\,bad\,\text{"}\right\} = \frac{e^{-5}\left(5\right)^{7}}{7!} = 0.104$$

$$P\left\{\underset{\sim}{X}=7 \,\middle|\, \underset{\sim}{L}=\text{"}\,dismal\,\text{"}\right\} = \frac{e^{-3}\left(3\right)^{7}}{7!} = 0.022$$

- ☐ **Then** $P\left\{\underset{\sim}{L}=\text{"}\,good\,\text{"} \,\middle|\, \underset{\sim}{X}=7\right\} =$

$$\frac{P\left\{\underset{\sim}{X}=7 \,\middle|\, \underset{\sim}{L}=\text{"}\,good\,\text{"}\right\}\cdot P\left\{\underset{\sim}{L}=\text{"}\,good\,\text{"}\right\}}{\left[\begin{array}{c} P\left\{\underset{\sim}{X}=7 \,\middle|\, \underset{\sim}{L}=\text{"}\,good\,\text{"}\right\}\cdot P\left\{\underset{\sim}{L}=\text{"}\,good\,\text{"}\right\}+P\left\{\underset{\sim}{X}=7 \,\middle|\, \underset{\sim}{L}=\text{"}\,bad\,\text{"}\right\} \\ \bullet P\left\{\underset{\sim}{L}=\text{"}bad\,\text{"}\right\}+P\left\{\underset{\sim}{X}=7 \,\middle|\, \underset{\sim}{L}=\text{"}dismal\,\text{"}\right\}\bullet P\left\{\underset{\sim}{L}=\text{"}\,dismal\,\text{"}\right\} \end{array}\right]}$$

EXAMPLE: SOFT PRETZELS

$$P\{\underset{\sim}{L} = \text{"}good\text{"} \mid \underset{\sim}{X} = 7\} = \frac{(0.09)(0.7)}{(0.09)(0.7)+(0.104)(0.2)+(0.022)(0.1)}$$

$$= 0.733$$

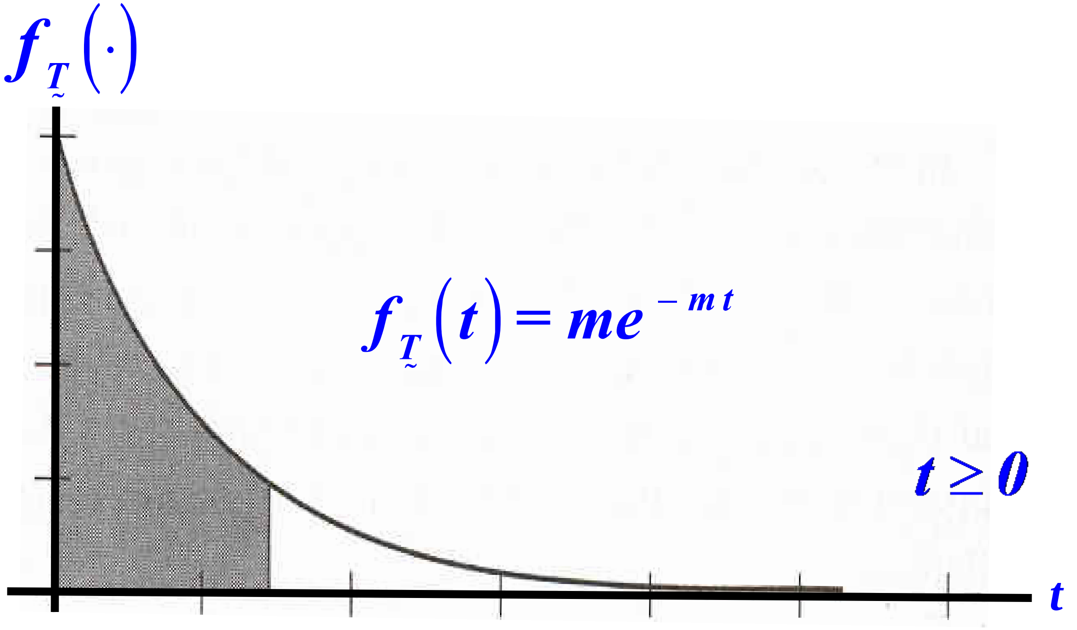
- ☐ **Similarly,**

$$P\left\{\underset{\sim}{L} = \text{"}bad\text{"} \quad \middle| \underset{\sim}{X} = 7\right\} = 0.242$$

$$P\left\{\underset{\sim}{L} = \text{"}dismal\text{"} \middle| \underset{\sim}{X} = 7\right\} = 1 - \left(0.733 + 0.242\right) = 0.025$$

EXPONENTIALLY DISTRIBUTED *r.v.*

- ☐ Unlike the discrete Poisson or the binomial distributed *r.v.*s, the exponentially distributed *r.v.* is continuous
- ☐ The density function has the form



EXPONENTIALLY DISTRIBUTED *r.v.*

- ☐ The exponentially distributed *r.v.* is related to the Poisson distribution
- ☐ Consider the Poisson distributed *r.v.* $\underset{\sim}{X}$ with $\underset{\sim}{X}$ representing the number of events in a given quantity of measure, *e.g.*, period of time
- ☐ We define $\underset{\sim}{T}$ to be the *r.v.* for the uncertain quantity we measure, *e.g.*, the time between 2 sequential events or the distance between 2 accidents

EXPONENTIALLY DISTRIBUTED *r.v.*

- **Then, $\underset{\sim}{T}$ has the exponential distribution with**

$$F_{\underset{\sim}{T}}\left(t\right) = P\left\{\underset{\sim}{T} \leq t\right\} = 1 - e^{-mt} ,$$

$$E\left\{\underset{\sim}{T}\right\} = \frac{1}{m} \quad and \quad var\left\{\underset{\sim}{T}\right\} = \frac{1}{m^{2}}$$

- **The exponentially distributed *r.v.* is completely specified by the parameter *m***

EXAMPLE: SOFT PRETZELS

- We know that it takes 3.5 minutes to bake a pretzel and we wish to determine the probability that the next customer will arrive after the pretzel baking is completed, i.e., $P\left\{ \underset{\sim}{T} > 3.5 \; minutes \right\}$
- We also are given that the location types are classified as being

EXAMPLE: SOFT PRETZELS

"*good*" *location* $\Leftrightarrow$ $m = 20$ *pretzels / hour*

"*bad*" *location* $\Leftrightarrow$ $m = 10$ *pretzels / hour*

"*dismal*" *location* $\Leftrightarrow$ $m = 6$ *pretzels / hour*

- ❑ **We compute the probability by conditioning on the location type and obtain**

EXAMPLE: SOFT PRETZELS

$$P\left\{\underset{\sim}{T} > 3.5\,minutes\right\} = P\left\{\underset{\sim}{T} > 3.5\,minutes \,\middle|\, m = 20\right\} \bullet P\left\{m = 20\right\} +$$

$$P\left\{\underset{\sim}{T} > 3.5\,minutes \,\middle|\, m = 10\right\} \bullet P\left\{m = 10\right\} +$$

$$P\left\{\underset{\sim}{T} > 3.5\,minutes \,\middle|\, m = 6\right\} \bullet P\left\{m = 6\right\}$$

$\equiv 0.0583\,hour$

- ❑ We evaluate

$$P\left\{\underset{\sim}{T} > 3.5m\right\} =$$

EXAMPLE: SOFT PRETZELS

$$e^{-0.0583(20)} P\{m=20\} + e^{-0.0583(10)} P\{m=10\} + e^{-0.0583(6)} P\{m=6\}$$

$$\textit{ex post probabilities}\left\{\begin{array}{l} P\{m=20\} = P\{\underset{\sim}{L} = \text{"}good\text{"} \mid \underset{\sim}{X} = 7\} = 0.733 \\ \\ P\{m=10\} = P\{\underset{\sim}{L} = \text{"}bad\text{"} \mid \underset{\sim}{X} = 7\} = 0.242 \\ \\ P\{m=6\} = P\{\underset{\sim}{L} = \text{"}dismal\text{"} \mid \underset{\sim}{X} = 7\} = 0.025 \end{array}\right.$$

EXAMPLE: SOFT PRETZELS

and so

$$P\left\{\underset{\sim}{T} > 3.5\ minutes\right\} = 0.3809$$

- ☐ Therefore,

$$P\left\{\underset{\sim}{T} \leq 3.5\ minutes\right\} = 1 - 0.3809 = 0.6191$$

and the interpretation is that the majority of the customers arrives before the pretzels are baked

THE NORMAL DISTRIBUTION

- ☐ The *normal* or *Gaussian* distribution is, by far, the most important probability distribution since the *Law of Large Numbers* implies that the distribution of many uncertain variables is governed by the *normal* distribution, or commonly known as the *bell curve*
- ☐ We consider a normally distributed *r.v.* $\underset{\sim}{Y}$

$$\underset{\sim}{Y} \sim \mathcal{N}\left(\mu,\sigma\right)$$

THE NORMAL DISTRIBUTION

- The density function is

mean

$$f_{\underset{\sim}{Y}}\left(y\right) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\left(\frac{1}{2}\frac{\left(y-\mu\right)^{2}}{\sigma^{2}}\right)}$$

variance

standard deviation

with $E\left\{\underset{\sim}{Y}\right\} = \mu \qquad and \qquad var\left\{\underset{\sim}{Y}\right\} = \sigma^{2}$

THE STANDARD NORMAL DISTRIBUTION

- Consider the *r.v.* $\underset{\sim}{Z}$ which has the standard normal distribution

$$\underset{\sim}{Z} \sim \mathcal{N}\left(0,1\right)$$

- The relationship between the *r.v.s* $\underset{\sim}{Y}$ and $\underset{\sim}{Z}$ is given by the affine relation:

$$\underset{\sim}{Z} = \frac{\underset{\sim}{Y} - \mu}{\sigma}$$

with

$$P\left\{\underset{\sim}{Y} \leq a\right\} = P\left\{\underset{\sim}{Z} \leq \left(a-\mu\right)/\sigma\right\}$$

THE STANDARD NORMAL DISTRIBUTION

- ☐ Note that

$$E\left\{\underset{\sim}{Z}\right\} = 0 \text{ and } var\left\{\underset{\sim}{Z}\right\} = 1$$

- ☐ In general, any value of the normal distribution is obtained from the *standard normal distribution* with the affine transformation

$$\underset{\sim}{Z} = \frac{\underset{\sim}{Y} - \mu}{\sigma}$$

EXAMPLE: QUALITY CONTROL

- **We consider a disk drive manufacturing process in which a particular machine produces a part used in the final assembly; the part must rigorously meet the width requirements within the interval [3.995, 4.005] *mm* ; else, the company incurs $10.40 in repair costs**
- **The machine is set to produce parts with the width of 4*mm*, but in reality, the width is a normally distributed *r.v.* $\underset{\sim}{W}$ with**

EXAMPLE: QUALITY CONTROL

$$\underset{\sim}{W} \sim \mathcal{N}\left(4,\sigma\right)$$

and

$$\sigma = f\left(speed\ of\ machine\right) = \begin{cases} 0.0019 & slow\ \ speed \\ 0.0026 & high\ \ speed \end{cases}$$

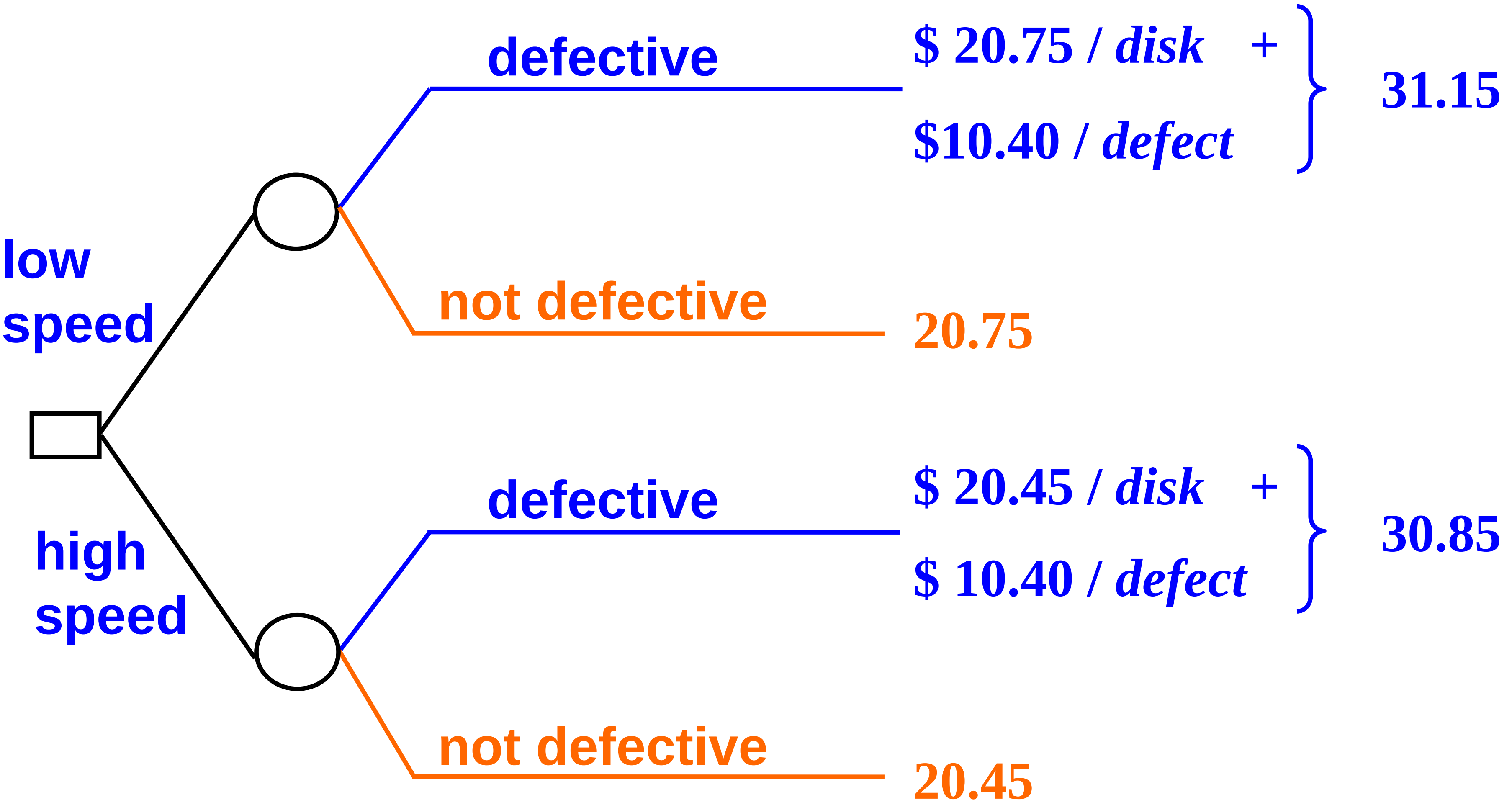
- ☐ **The respective costs in *\$* of the disk drive are**

20.75	*slow speed*
20.45	*high speed*

EXAMPLE: QUALITY CONTROL

- We may interpret the cost data to imply that more disks can be produced at lesser costs at the high speed
- The problem is to select the machine speed to obtain the more cost effective result
- A decision tree is useful in the analysis of the situation

EXAMPLE: QUALITY CONTROL



- **We evaluate the probability of each outcome**

LOW – SPEED PROBABILITY EVALUATION

$$P\left\{ \textit{defective disk is produced} \right\} =$$

$$P\left\{ \underset{\sim}{W} < 3.995 \textit{ or } \underset{\sim}{W} > 4.005 \right\} =$$

$$1 - P\left\{ 3.995 \leq \underset{\sim}{W} \leq 4.005 \right\} =$$

$$\underset{\sim}{Z} = \frac{\underset{\sim}{W} - 4}{0.0019}$$

$$1 - P\left\{ \frac{3.995 - 4}{0.0019} \leq \underset{\sim}{Z} \leq \frac{4.005 - 4}{0.0019} \right\} =$$

LOW – SPEED PROBABILITY EVALUATION

$$1-P\left\{-2.63\le \underset{\sim}{Z}\le 2.63\right\} =$$

$$1-\left[\underbrace{\underbrace{P\left\{\underset{\sim}{Z}\le 2.63\right\}}_{0.9957}-\underbrace{P\left\{\underset{\sim}{Z}\le -2.63\right\}}_{0.0043}}_{0.9914}\right] = \mathbf{0.0086}$$

HIGH – SPEED PROBABILITY EVALUATION

$$P\left\{ \textit{defective disk is produced} \right\} \qquad =$$

$$P\left\{ \underset{\sim}{W} < 3.995 \textit{ or } \underset{\sim}{W} > 4.005 \right\} \qquad =$$

$$1 - P\left\{ 3.995 \leq \underset{\sim}{W} \leq 4.005 \right\} \qquad =$$

$$\underset{\sim}{Z} = \frac{\underset{\sim}{W} - 4}{0.0026}$$

$$1 - P\left\{ \frac{3.995 - 4}{0.0026} \leq \underset{\sim}{Z} \leq \frac{4.005 - 4}{0.0026} \right\} \qquad =$$

HIGH – SPEED PROBABILITY EVALUATION

$$1 - P\left\{-1.92 \le \underset{\sim}{Z} \le 1.92\right\} =$$

$$1 - \left[\underbrace{\underbrace{P\left\{\underset{\sim}{Z} \le 1.92\right\}}_{0.9726} - \underbrace{P\left\{\underset{\sim}{Z} \le -1.92\right\}}_{0.0274}}_{0.9452} \right] = \mathbf{0.0548}$$

MEAN VALUE EVALUATION

- We next evaluate the mean cost per disk

$$E\left\{cost/disk \,\middle|\, low\ speed\right\} = (0.9914)(20.75) + (0.0086)(31.15)$$

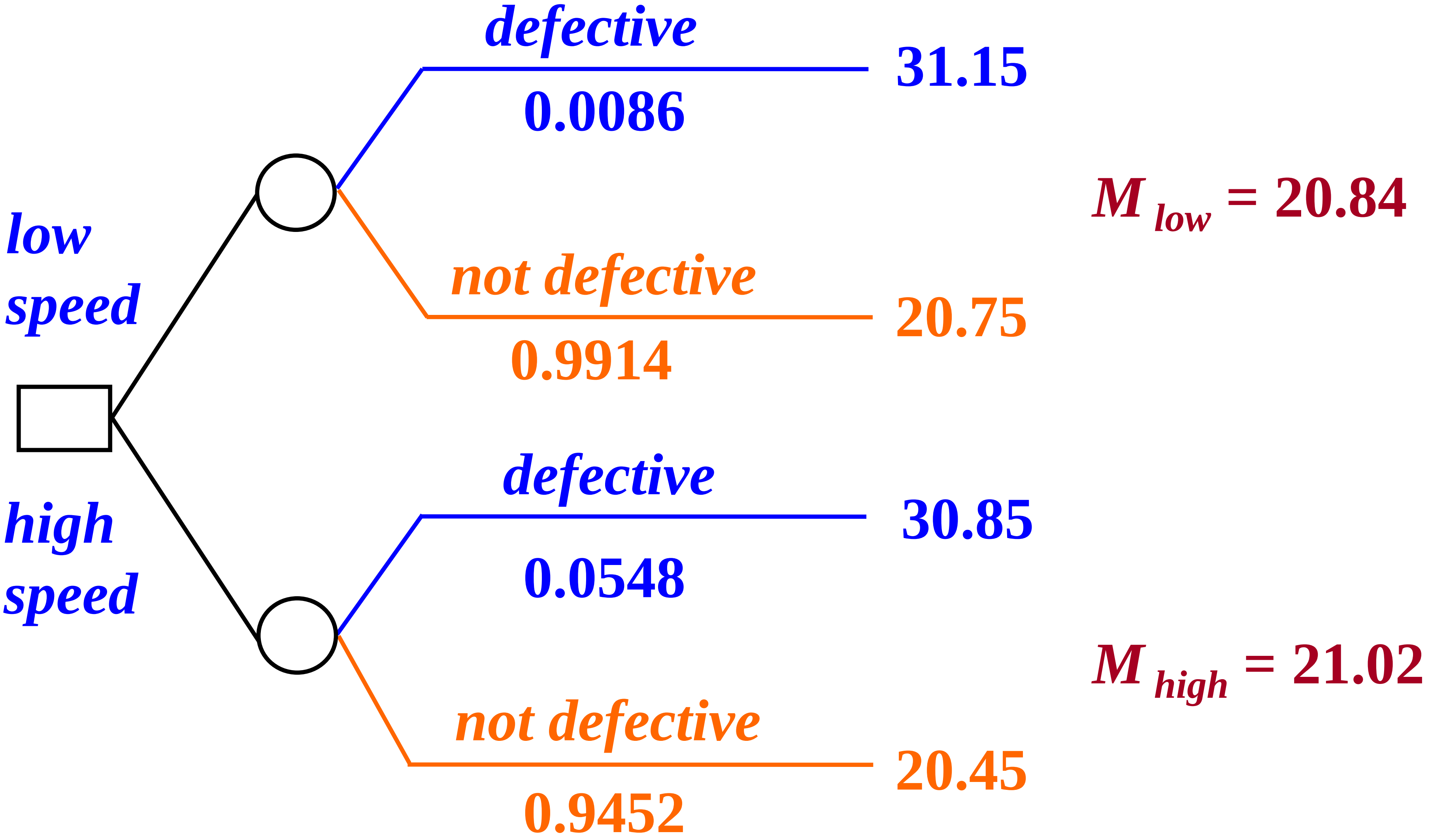
$$= 20.84$$

$$E\left\{cost/disk \,\middle|\, high\ speed\right\} = (0.9452)(20.45) + (0.0548)(30.85)$$

$$= 21.02$$

- We summarize the information in the decision tree

EXAMPLE: QUALITY CONTROL



$M_{low} = 20.84$

$M_{high} = 21.02$

CASE STUDY: OVERBOOKING

- ❑ *M Airlines* has a commuter plane capable of flying 16 passengers
- ❑ The plane is used on a route for which *M Airlines* charges $ 225
- ❑ The airliner's cost structure is based on

the fixed costs for each flight	$ 900
the variable costs/passenger	$ 100
the "no-show" rate	4 %

CASE STUDY – OVERBOOKING

- The refund policy is that unused tickets are refunded only if a reservation is cancelled 24 *h* before the scheduled departure
- The overbooking policy pays *$* 100 as an incentive to each bumped passenger and refunds the ticket
- The decision required is to determine how many reservations should the airliner sell on this plane

SAMPLE CALCULATION FOR SELLING 18 RESERVATIONS

total revenues: $\underset{\sim}{R} = 225 \bullet 18 = 4,050$

passenger fixed and variable costs:

$$C_1 = 900 + 100 \bullet min\{number\ of\ "shows", 16\}\$$$

bumping costs:

$$C_2 = (225 + 100) \bullet max\{0, number\ of\ "shows" - 16\}\$$$

refunds to customers

total costs: $\underset{\sim}{C} = C_1 + C_2$

CASE STUDY: OVERBOOKING

- ☐ **We evaluate**

$$P\left\{ no.\ of\ "shows" > 16 \ \middle|\ reservations\ sold = 18 \right\}$$

- ☐ **We assume that each reservation is a** *r.v.* $\underset{\sim}{P}_i$ **:**

$$\underset{\sim}{P}_i = \begin{cases} 1 & passenger\ \ i\ \ is\ a\ "show"\ \ with\ \ prob.\ 0.96 \\ 0 & passenger\ \ i\ \ is\ a\ "no\ show"\ \ with\ \ prob.\ 0.04 \end{cases}$$

CASE STUDY: OVERBOOKING

- ☐ **If reservations sold = 18 , then we need to**

  evaluate

$$P\left\{ \sum_{i=1}^{18} \underset{\sim}{P}_i > 16 \,\middle|\, 18 \; \textit{reservations} \right\}$$

- ☐ **We first evaluate**

$$P\left\{ \sum_{i=1}^{17} \underset{\sim}{P}_i > 16 \,\middle|\, 17 \; \textit{res} \right\} = P\left\{ \sum_{i=1}^{17} \underset{\sim}{P}_i \geq 17 \,\middle|\, 17 \; \textit{res} \right\} = P\left\{ \sum_{i=1}^{17} \underset{\sim}{P}_i = 17 \,\middle|\, 17 \; \textit{res} \right\}$$

binomial *r.v.* with $p = 0.96$

CASE STUDY: OVERBOOKING

$$\boxed{= (.96)^{17} (.04)^{0}} \longleftarrow 0.4996$$

- Then,

$$P\left\{ \sum_{i=1}^{18} \underset{\sim}{P}_{i} > 16 \,\middle|\, 18\, res \right\} = P\left\{ \sum_{i=1}^{18} \underset{\sim}{P}_{i} \geq 17 \,\middle|\, 18\, res \right\} =$$

$$\underbrace{P\left\{ \sum_{i=1}^{18} \underset{\sim}{P}_{i} = 17 \,\middle|\, 18\, res \right\}}_{18(.4996)(.04)} + \underbrace{P\left\{ \sum_{i=1}^{18} \underset{\sim}{P}_{i} = 18 \,\middle|\, 18\, res \right\}}_{(.4996)(.96)} = 0.8359$$

CASE STUDY: OVERBOOKING

- ☐ If reservations sold = 19 , then we can compute and show that

$$P\left\{ \sum_{i=1}^{19} \underset{\sim}{P}_i > 16 \,\middle|\, 19\, res \right\} = .9616$$

- ☐ We next consider the profit *r.v.* $\underset{\sim}{\pi}$, where,

$$\underset{\sim}{\pi} = \underset{\sim}{R} - \underset{\sim}{C} = \underset{\sim}{R} - \left(C_1 + C_2 \right)$$

and evaluate $E\left\{ \underset{\sim}{\pi} \right\}$ for different values of reser– vations sold

CASE STUDY: OVERBOOKING

- ☐ For reservations = 16

$$E\{\underset{\sim}{R}\} = (16)(225) = 3,600$$

$$E\{C_1\} = 900 + 100\sum_{n=0}^{16} nP\left\{\sum_{i=1}^{16}\underset{\sim}{P}_i = n\right\}$$

$$= 900 + 100\ \underbrace{E\left\{\sum_{n=1}^{16}\underset{\sim}{P}_i\right\}}_{(16)(.96)=15.36}$$

binomial distribution

$$= 900 + 1,536$$

CASE STUDY: OVERBOOKING

$$= 2,436;$$

also,

$$E\left\{C_2\right\} = \left(225 + 100\right) max \left\{ 0, \sum_{i=1}^{16} \underset{\sim}{P}_i - 16 \right\} = 0$$

and so

$$E\left\{\underset{\sim}{\pi} \middle| 16\ res\right\} = E\left\{\underset{\sim}{R}\right\} - E\left\{\underset{\sim}{C}\right\}$$

$$= E\left\{\underset{\sim}{R}\right\} - E\left\{C_1 + C_2\right\}$$

$$= 3,600 - 2,436$$

$$= 1,164$$

CASE STUDY: OVERBOOKING

- ☐ **For reservations = 17**

$$E\left\{\underset{\sim}{R}\right\} = (17)(225) = 3{,}825$$

$$E\left\{C_1\right\} = 900 + 100\sum_{n=0}^{16} nP\left\{\sum_{i=1}^{17} \underset{\sim}{P}_i = n\right\} + 100.16 \bullet P\left\{\sum_{i=1}^{17} \underset{\sim}{P}_i = 17\right\}$$

$$= 900 + 782.70 + 799.34$$

$$= 2{,}482.04$$

CASE STUDY: OVERBOOKING

also,

$$E\left\{C_2\right\} = 325P\left\{\sum_{i=1}^{17} \underset{\sim}{P}_i = 17\right\}$$

$$= 325\left(0.4996\right)$$

$$= 162.37$$

and so

$$E\left\{\underset{\sim}{\pi} \middle| 17\ res\right\} = 3{,}825 - 2{,}482.04 - 162.37$$

$$= 1{,}180.59 > 1{,}164$$

CASE STUDY: OVERBOOKING

- For reservations = 18

$$E\left\{\underset{\sim}{R}\right\} = (18)(225) = 4,050$$

$$E\left\{C_1\right\} = 900 + 100\sum_{n=0}^{16} nP\left\{\sum_{i=1}^{18} \underset{\sim}{P}_i = n\right\} + 1,600 \cdot P\left\{\sum_{i=1}^{18} \underset{\sim}{P}_i > 16\right\}$$

$$= 900 + 253.22 + 1,342.89$$

$$= 2,496.11$$

CASE STUDY: OVERBOOKING

$$E\left\{C_2\right\} = 325\,\underbrace{P\left\{\sum_{i=1}^{18} \underset{\sim}{P}_i = 17\right\}}_{.3597} + 650\,\underbrace{P\left\{\sum_{i=1}^{18} \underset{\sim}{P}_i = 18\right\}}_{.4796} = 428.65$$

and

$$E\left\{\underset{\sim}{\pi} \,\middle|\, 18\ res\right\} = 4{,}050 - 2{,}496.11 - 428.65$$

$$= 1{,}125.24$$

$$< 1{,}180.59$$

CASE STUDY: OVERBOOKING

- ☐ **We can show that for reservations = 19**

$$E\left\{\underset{\sim}{\pi} \mid 19\ res\right\} < 1180.59$$

- ☐ **We conclude that the profits are maximized for reservations = 17 and so any overbooking over that number results in lower profits**